Linear structures of permutation groups over vector spaces

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Linear structures

Let $n, m \geq 1$,

$$\Pi_{\alpha,\gamma} = \left\{ \pi : V_n \rightarrow V_m \mid \beta^{\pi} + (\beta + \alpha)^{\pi} = \gamma,\ \forall \beta \in V_n \right\},$$

$$\alpha \in V_n / \vec{0}, \gamma \in V_m / \vec{0}.$$

The mapping $\pi : V_n \rightarrow V_m$ has **nonzero linear structures** if there exists $\alpha \in V_n / \vec{0}$ such that

$$\left| \Pi_{\alpha,\gamma} \right| = 2^n.$$

For *n*=*m* we describe mappings with linear structures using groups.

It is known that the mapping $\pi : V_n \to V_n$ has nonzero linear structures if there exists a nonsingular $n \times n$ matrix B such that

$$\pi(Bx) = h(x_1, \ldots, x_l) + g(x_{l+1}, \ldots, x_n),$$

where h is a linear mapping, g is a mapping without linear structures.

Permutation groups with linear structures

Let

$$\Pi_{W,W} = \left\{ \pi \in S(V_n) \mid \beta^{\pi} + (\beta + \alpha)^{\pi} \in W,\ \forall \beta \in V_n, \forall \alpha \in W \right\},$$

where W is a subspace of V_n.

$$\Pi_{W,W} = \Pi_{\alpha,\alpha} \text{ if } W = \{\vec{0}, \alpha\}.$$

Theorem 1. For any subspace $W \le V_n$, $\dim W = t \in \{\overline{1,n}\}$, $\Pi_{W,W} = S_{2^t} wr S_{2^{n-t}}$ is an imprimitive permutation group from $S(V_n)$.

$$\pi : \beta + W \to \beta^{\pi} + W \text{ for any } \pi \in \Pi_{W,W}$$

Proposition 2. For any set $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq V_n$, $k \in \overline{\{1, 2^n\}}$, $\dim < \alpha_1, \alpha_2, \ldots, \alpha_k >= t \geq 1$, the following holds

$$\bigcap_{i=1}^{k} \Pi_{\alpha_i, \alpha_i} = \underbrace{S_2 wr \ldots wr S_2 wr}_{t} S_{2^{m-t}}.$$

Permutation groups with linear structures

Theorem 3. Let $\alpha, \gamma \in V_n \setminus \vec{0}$, h be an invertible linear mapping from $\Pi_{\alpha,\gamma}$. Then

$$\Pi_{\alpha,\gamma} = \Pi_{\alpha,\alpha} h = \left(S_2 wr S_{2^{m-1}} \right) h .$$

If h be an invertible linear mapping from $\Pi_{\gamma,\alpha}$. Then

$$\Pi_{\alpha,\gamma} = h \Pi_{\gamma,\gamma} = h \left(S_2 wr S_{2^{m-1}} \right).$$

Proposition 4. Let $\alpha, \gamma \in V_n \setminus \vec{0}$. Then

$$\left|\Pi_{\alpha,\gamma}\right| = 2^{2^{m-1}} \cdot (2^{m-1}!).$$

The number of permutations from $S(V_n)$ with the linear structure α is equal to

$$(2^m - 1) \cdot 2^{2^{m-1}} \cdot (2^{m-1}!).$$